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Abstract

Late ancient Platonists and Aristotelians describe the method of reasoning to first principles as "analysis." This is a metaphor from geometrical practice. How far back were philosophers taking geometric analysis as a model for philosophy, and what work did they mean this model to do? After giving a logical description of analysis in geometry, and arguing that the standard (not entirely accurate) late ancient logical description of analysis was already familiar in the time of Plato and Aristotle, I argue that Plato, in the second geometrical passage of the *Meno* (86e4-87b2), is taking analysis as a model for one kind of philosophical reasoning, and I explore the advantages and limits of this model for philosophical discovery, and in particular for how first principles can be discovered, without circularity, by argument.

I

Aristotle cites Plato as asking whether, at any given stage in an argument, "we are on the way to the principles or from the principles" (NE I,4 1095a32-3). The word "principle," ἀρχή, means literally "beginning." Greek philosophers use the word to mean whatever is prior to everything else. This might be straightforward temporal priority – for the pre-Socratics, the ἀρχαί are whatever there was before the ordered universe was formed out of them – but Plato and Aristotle extend the word to more abstract senses of priority. Philosophers before Plato had assumed that the beginning of things is also the right beginning for our argument or discourse about the things; Plato and Aristotle are saying, by contrast, that there are two stages of argument, first *to* the ἀρχαί (contrary to the "natural" order of the things) and then *from* the ἀρχαί (following the natural order). Plato's point is that, when we begin an argument, we are not immediately in a position to grasp the ἀρχαί, but must somehow reason back to them from something more immediately evident. As Aristotle puts it, we must begin with "the things that are better known to us" and reason to "the things that are better known by nature" and are prior by nature: the goal is to make the things that are better known by nature also better known to us, so that we can use them as a starting-point of argument to gain scientific knowledge of the things that are derived from these ἀρχαί.

Late ancient philosophers (middle- and neo-Platonists, and the Peripatetic Alexander of Aphrodisias) are very interested in this process of reasoning back to first principles. They call it "analysis," and contrast it with "synthesis" as reasoning *from* the principles; and something like this description of analytical reasoning remains familiar to us.[1] But this description is very problematic, and my goal here is to begin to get clear on the problems that it involves.

To describe Plato's or Aristotle's procedure of reasoning to principles as "analysis" is to apply a metaphor from geometry: "analysis" is the name for a definite geometrical procedure, and the neo-Platonists are conscious that they are speaking metaphorically in extending the term to philosophy. The neo-Platonists were (at their best) mathematically well-educated people, and the study of analysis (what Pappus calls the ἀναλυόμενος τόπος) is the key to the non-elementary part of geometry, in Pappus' words "a special resource that was prepared, after the discovery of the common Elements, for those who want to acquire in geometry a power of solving problems set to them".[2] The neo-Platonists want to claim a connection between this geometrical procedure and the philosophical method of arguing to first principles. Proclus actually says that Plato "taught" geometrical analysis to the geometer Leodamas (*In Euclidem* p. 211, cp. p. 66), apparently implying that Plato invented the method of analysis and passed it on to the mathematicians.[3] Certainly Plato did not invent analysis; there is good reason to think that the method was used by Hippocrates of Chios as early as 430 BC; Proclus routinely credits Plato with inventing any mathematical idea or proposition that is mentioned in a Platonic dialogue,[4] and we need not take these ascriptions too seriously. But it is worth thinking about why Proclus and other Platonists would want to claim the method of analysis for Plato. The method of analysis had enormous prestige, in antiquity and down to the days of Descartes and Fermat,

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[1] Many of the relevant texts of late ancient philosophers (from Alcinous through the sixth century AD) are collected and discussed by Donald Morrison in a work-in-progress, which I have used with profit.

[2] From the beginning of Pappus' *Collection* VII, Jones' translation modified (text and translation from Pappus of Alexandria, *Book 7 of the Collection*, edited with translation and commentary by Alexander Jones, 2 vols., New York-Berlin-Heidelberg, 1986).

[3] Diogenes Laertius in his *Life of Plato* (DL III,24) attributes the same report to the early second century AD Academic Favorinus; it must have become a commonplace of the Platonic school.

[4] As noted by Wilbur Knorr, *The Evolution of the Euclidean Elements* (Dordrecht, 1975), p. 6.

because it was seen as the basic method of mathematical *discovery*: not simply a way for a student to discover and assimilate for himself propositions already known to his teachers, but also a way for a mature geometer to discover previously unknown propositions. While analysis is a method with clear rules for step-by-step work (though it is not a *mechanical* method – the geometer must apply the rules intelligently in order to succeed), it terminates when something unpredictably "clicks"; then, if and when this happens, the geometer must again proceed methodically (again, not mechanically) by the method of synthesis, to confirm what has been discovered by analysis; if this succeeds, then the newly discovered proposition may be presented with a demonstration in the usual highly stylized form given in the classic Greek mathematical texts. Since it is obvious that the propositions of, for instance, Euclid's *Elements* or Apollonius' *Conics* were not first discovered by means of the demonstrations that are now used to justify them, it is natural to ask how else they were discovered; and in many cases it is natural to suspect that they were first discovered by analysis. Indeed, Archimedes actually gives us the analysis as well as the synthetic demonstration for several propositions of *On the Sphere and the Cylinder* II, and so does Apollonius for several propositions of the *Conics*, and for all the propositions of the *Cutting-off of a Ratio* (extant in an Arabic translation): these are by far the earliest extant examples of analysis, and they show that much later accounts of analysis, such as Pappus', do faithfully reflect geometrical practice at least as far back as the third century BC.[5] So it was natural for the philosophers

[5] Archimedes *On the Sphere and the Cylinder* Book II,1 and 3-7; Apollonius *Conics* II,44-47 and 49-51; Jones discusses the *Cutting-off of a Ratio*, op. cit., pp. 510-12 and translates some sections of it pp. 606-19. All of these analyses are of problems rather than theorems. There are theoretical analyses in the manuscripts of Euclid *Elements* XIII,1-5, but these are agreed to be post-Euclidean, and their origin and date are uncertain (the text is printed in Heiberg's Teubner *Elements* in an appendix, v. 4 pp. 364-77). On the history of analysis, as opposed to the history of descriptions of analysis by mathematicians or philosophers, there is useful material in Wilbur Knorr, *The Ancient Tradition of Geometric Problems* (Boston, 1986) and in Richard Ferrier's introduction to *The Data of Euclid*, translated by George McDowell and Merle Sokolik (Baltimore, 1993); McDowell and Sokolik also translate one of the extant analyses from Apollonius' *Conics*, showing how it makes use of the *Data*. Also Jaakko Hintikka and Unto Remes in *The Method of Analysis* (Dordrecht, 1974), in addition to discussing the logical structure of analysis from a modern point of view, give useful annotated versions of two analyses in Pappus, pp. 22-6 and pp. 52-3. Finally, Alexander Jones' translation and commentary on Pappus *Collection* Book VII, cited above, are very useful.

to regard analysis as the living core of Greek mathematical thought, what the geometers did by themselves and taught to their students, while most of the works they publish and make available to people outside the school are only the dead husks. If analysis is such a powerful source of insight in mathematics, it is natural for the philosophers to hope to find something like it in philosophy.

At least from the second century AD on, the philosophers were interested in taking geometrical analysis as a model for philosophy: the hope was that the proved success of the method in mathematics would rub off on the philosophers, or, more cynically, that the prestige of the method in mathematics would help to justify what the philosophers were already doing.[6] What I am interested in is whether Plato, in the fourth century BC, was already taking this kind of interest in the method of analysis. Proclus likes to think of Plato the philosopher as giving directions to the mathematicians about what problems to work on, and even as teaching them mathematical methods, but it is more fruitful to ask what philosophical inspiration Plato may have taken from the mathematicians: for it is certainly clear that Plato was an enthusiast for mathematical training and that he "everywhere tries to arouse admiration for mathematics among students of philosophy" (Proclus *In Euclidem* p. 66).[7] Plato and his contemporaries do not use "analysis," as the neo-Platonists will, as a general term for all reasoning to first principles in philosophy as well as mathematics. Indeed, Plato never uses the word "analysis" (or the verb ἀναλύειν) at all. But that does not show that Plato and others in the Academy may not have thought about or alluded to geometrical analysis, or taken it as a model in their own reasoning. I will argue that Plato does, at least once, allude to geometrical analysis, and that he at least experimented with taking it as a model for philosophical reasoning.

First I should say what the method of analysis was. It is standard to start by commenting on the late ancient definitions or descriptions of analysis, since we have nothing like a definition of analysis before about the first century AD (probably the earliest are Heron and Alcinous; the only extended description is in Pappus, probably third century).[8] Unfortunately, these descriptions of analysis are unclear and sometimes

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[6] This history will be traced in Don Morrison's work-in-progress cited above.

[7] For a sceptical treatment of the image of Plato as "research director" of mathematics in the Academy, see now Leonid Zhmud, "Plato as 'Architect of Science,'" in *Phronesis*, v. 43 (1998), pp. 211-44.

[8] The Heron text, short and probably not improved in translation, is cited by Nairîzî

misleading at crucial points, so they need a fair amount of commentary. Fortunately, we are not dependent on these "official" descriptions, and (despite the impression one might get from the scholarly literature) there is no real doubt about what analysis was – there are the extant texts of Archimedes, Apollonius and Pappus practicing analysis, and this is not a lost art, but something one can easily train oneself to do on the ancient model; and once we master the practice, we can understand the official

("Anaritius"); a medieval Latin translation is printed in the Teubner Euclid, v. 5, p. 89, lines 13-21. Alcinous discusses analysis in *Didaskalikos* chapter 5, sections 4-6 (his "second type of analysis" is the relevant one); also the interpolated analyses of Euclid *Elements* XIII,1-5, cited in a previous note, begin with brief definitions of analysis and synthesis (though the definition of synthesis is corrupt), which may conceivably be the earliest extant. The only account that might actually be helpful is Pappus': "What is called the Domain of Analysis, my son Hermodorus, is, in sum, a special resource that was prepared, after the discovery of the common Elements, for those who want to acquire in geometry a power of solving problems set to them; and it is useful for this alone. It was written by three men – Euclid the Elementarist, Apollonius of Perga, and Aristaeus the elder – and it proceeds by the methods of analysis and synthesis. Now analysis is the path from what one is seeking, as if it were established [or agreed], through the things that follow, to something that is established by synthesis. For in analysis we assume what is sought as if it has been achieved, and look for the thing from which it comes about, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call such a method 'analysis,' that is, ἀνάπαλιν λύσις ['solution backwards']. In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and by arranging in their natural order as antecedents what were consequents in the analysis, and by putting them together, we reach the goal of the construction of what was sought; and we call this 'synthesis.' There are two kinds of analysis: one of them seeks after truth, and is called 'theoretic' [or 'theorematic'], while the other tries to furnish something that has been prescribed [for us to construct], and is called 'problematic.' In the theoretic kind, we assume what is sought as being, i.e. as true, and then proceed, through what follows as true-and-being according to the assumption, to something that is [already] agreed: if it is agreed to be *true*, what was sought will also be true, and its proof is the reverse of the analysis; but if we encounter something agreed to be *false*, then what was sought will also be false. In the problematic kind, we assume the thing prescribed as if known, and then, proceed through what follows as true, to something that is [already] agreed: if it is agreed to be possible and furnishable (what the mathematicians call 'given'), the thing prescribed will also be possible, and again the proof will be the reverse of the analysis; but if we encounter something agreed to be impossible, the problem too will be impossible. Διορισμός is adding a condition [reading προσδιαστολή for προδιαστολή] on when, how, and in how many ways the problem will be possible. So much, then, on analysis and synthesis." (Pappus, *Collection* VII,1-2, Jones' translation modified).

descriptions, while recognizing their imprecisions.[9] While analysis is not terribly mysterious, it is difficult to give a precise logical account of it; and given the general failure of the Greeks at describing the logical structure of mathematical reasoning, it is not surprising that their descriptions of the logic of analysis can be improved on. A basically correct logical description has been given by Hintikka and Remes (along with, unfortunately, much that is incorrect), and I will make use of their work, as well as of the extant ancient examples and descriptions of analysis.

The most common account of analysis, both in the ancient sources and in modern reconstructions, goes something like this. We are trying to prove a proposition P, the ζητούμενον or thing-sought. As a heuristic toward finding a proof, we assume the ζητούμενον P as if it were known to be true, and then draw inferences from this assumption; the analysis terminates – and something "clicks" – when we derive either a proposition known to be true (from the principles of geometry and from theorems we have already proved), or else a proposition known to be false. If we can infer from P to a proposition known to be false, then we have proved $\neg P$ by *reductio ad absurdum*. If we have inferred from P to a proposition R that we know to be true, then we can try to *reverse* each step of the derivation of R from P: if this succeeds, then the proof of R, together with the derivation of P from R, give us a proof of P. Of course, there is no *guarantee* that the analysis (i.e. the derivation of R from P) is reversible: but it does very frequently happen that steps of geometrical arguments are reversible (i.e. that if a step $P \rightarrow Q$ is legitimate, so is $Q \rightarrow P$; e.g. "if triangle ABC is isosceles, it has equal base angles", and also "if triangle ABC has equal base angles, it is isosceles"), and in carrying out the analysis intelligently we will try to avoid obviously non-reversible steps (e.g. "if triangle ABC is isosceles, its angles are equal to two right angles"). So although an analysis leading to a positive result does not guarantee that the ζητούμενον can be proved, it may still be heuristically useful, since it constructs a plausible outline for a proof, and we can then try to fill in the steps. Some ancient (and modern) writers, bothered by the logical gap between a successful analysis of P and a successful proof of P, say instead that analysis begins from the ζητούμενον P and proceeds (not to propositions that follow from P but) to "propositions from which the ζητούμενον

[9] Nineteenth-century geometry texts often gave instructions for how to carry out analyses and syntheses – these instructions are logically inexact, but show that the authors, and the teachers and students who used their books, did habitually carry out the practice of analysis.

would follow," so that as soon as we reach a proposition known to be true, we would have a guaranteed proof of P. But it is historically clear that analysis was always a deductive procedure starting with the ζητούμενον, and this is also much more practicable and heuristically useful. The only legitimate sense in which in analysis we are looking for "propositions from which the ζητούμενον would follow" rather than "propositions which follow from the ζητούμενον" is that we are looking for propositions which, *in the completed demonstration*, will be prior to P, and so which, in some vague "causal" sense, may be seen as "naturally" prior to P.

However, as Hintikka and Remes recognized, this standard account of analysis is logically very imprecise, and its imprecisions make it hard to see why analysis would be heuristically valuable. While analysis looks for a proof of a proposition by assuming the ζητούμενον as if it were known and drawing inferences from it, it is a serious mistake to identify the ζητούμενον with the proposition we are trying to prove. To begin with, Greek mathematical texts contain two kinds of propositions, *theorems* and *problems*, and analysis may be seeking a proof of either. Only theorems are what *we* would call propositions: a theorem is a statement asserting that all figures of a given class have some particular property, while a *problem* is a challenge to construct a figure having certain prescribed properties (and/or certain prescribed relations to a given figure). While the enunciation of a theorem is a complete sentence, the enunciation of a problem is an infinitive phrase (e.g. "to inscribe in a given circle a triangle similar to a given triangle" or "to construct an equilateral and equiangular pentagon"). Pappus distinguishes accordingly between "theoretic" analysis (analysis of theorems) and "problematic" analysis (analysis of problems). In problematic analysis the ζητούμενον is not a proposition at all, but rather an object, the figure we are trying to construct: we assume the desired figure *as if* its size, shape and position were known, and make constructions out of it (and draw inferences about the figures we construct, from the assumption that the ζητούμενον had the prescribed properties), until "something clicks" and we construct a figure whose size, shape and position we recognize that we can determine from the givens of the problem alone (together with the principles of geometry and with propositions we have already proved), independently of our assumption about the ζητούμενον. Having reached this point, we then try to reverse the construction and the accompanying inferences to produce a construction of the ζητούμενον and a proof that it does indeed have the prescribed properties. The ancient general accounts of analysis which speak of assuming the ζητούμενον as if it were known, inferring to something independently known, and then reversing the reasoning to demonstrate the ζητούμενον, are deliberately vague enough to include this case; we miss problematic analysis, and thus badly misrepresent the analytic method, if we identify the ζητούμενον with the proposition we are trying to prove.

In fact, even in the case of theoretical analysis, the ζητούμενον is generally not the proposition to be proved. A typical theorem can be represented as $\forall x\ (Px \rightarrow Qx)$[10] (e.g. "for any triangle ABC, if AB = AC, then $\angle ABC = \angle ACB$"). In analyzing this theorem, the ζητούμενον will not be the proposition $\forall x\ (Px \rightarrow Qx)$, but rather the proposition Qx. We take as given an arbitrary x such that Px; and then we *also* assume the ζητούμενον Qx *as if* it were known to be true; we then draw inferences from the ζητούμενον Qx, making use of the "given" Px as well as of the principles of geometry and of other propositions already proved. The analysis terminates when we infer a proposition Rx whose truth value we can determine independently of the ζητούμενον Qx. If Rx is known to be true, from Px as well as the principles of geometry and other propositions already proved, then we can try to reverse the analysis. We could describe this reversal as turning the derivation of Rx from Qx into a derivation of Qx from Rx; but since we used the fact that Px in deriving Rx from Qx, and since we will also have to use Px in reversing the analysis to derive Qx from Rx, it is more accurate to say that we are turning a derivation of $Rx \cap Px$ from $Qx \cap Px$ into a derivation of $Qx \cap Px$ from $Rx \cap Px$. If we can do this, we have a proof of the theorem $\forall x\ (Px \rightarrow Qx)$: first assume an arbitrary x such that Px, then infer from Px to Rx, then infer from $Px \cap Rx$ to Qx, then conclude that $\forall x\ (Px \rightarrow Qx)$.[11] Or the analysis could have a negative result: either we could infer from $Qx \cap Px$ to a proposition Rx that we know to contradict Px (in which case we have proved $\forall x\ (Px \rightarrow \neg Qx)$, so $\forall x\ (Px \rightarrow Qx)$ is false unless $\forall x\ \neg Px$); or we might recognize that Rx is true for some but not all x such that Px, in which case we know that $\forall x\ (Px \rightarrow Qx)$ is false, but we can conjecture that the analysis is reversible and that $\forall x\ (Px \cap Rx \rightarrow Qx)$, with the extra condition added, will be true. As Hintikka and Remes point out, the analysis is not simply working "backward" from the ζητούμενον Qx to infer the "beginning" Px: rather,

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[10] When I say "$\forall x$", this means a block of universal quantifiers, possibly more than one (but it is important that there are no *existential* quantifiers in Greek theorems); similarly, "Px" may really be a relational expression "Pxyz".

[11] The argument will depend on a natural-deduction step, but so do all proofs in Greek geometry, since they all work by proving the instance of the proposition set out in the ἔκθεσις and διορισμός, and then inferring to the universal proposition.

the analysis starts by assuming *both* ends together, and this additional logical power helps to explain why we are likely to be able to infer something that we can recognize as true (or as false): whereas, if we were simply arguing "backwards" from Qx to Px, it is hard to see why this should be heuristically any more useful than arguing "forwards."[12]

We can try to give a similar logical description of problems and of problematic analysis. This should come with a warning label, since, in the case of problems more clearly than in the case of theorems, we are in conflict with the Greek conception of these propositions when we represent them in the notation of the predicate calculus. To every Greek problem there is a corresponding proposition with the logical form $\forall x\ (Px \rightarrow \exists y\ Qxy)$ [or in prenex normal form $\forall x \exists y\ (Px \rightarrow Qxy)$] – so the problem "to inscribe in a given circle a triangle similar to a given triangle" can be rewritten as "for any circle x and any triangle x', there is a triangle y such that y is inscribed in x and y is similar to x'". Any demonstration of the problem also demonstrates this universal-existential proposition; but the only acceptable demonstrations of the problem are *constructive* demonstrations, that is, procedures that show how to construct the ζητούμενον y from any given x such that Px, accompanied by a proof that Qxy.[13] Greek theorems, as opposed to problems, *never* involve existential quantifiers; universal-existential propositions appear in Greek mathematics *only* in the guise of problems.[14] I am not saying that Greek geometers *identified* the

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[12] With the above account compare Hintikka and Remes, esp. pp. 31-9.

[13] However, the proof might not satisfy the scruples of a modern constructivist, since it might rely on the law of the excluded middle.

[14] This needs some qualification, since there are some propositions, sometimes called "porisms" rather than "problems," which are phrased as a challenge, not to *construct* something, but to *find* something (because it is a point or an abstract magnitude or number, none of which are properly "constructed," or because it has already implicitly been constructed in the ἔκθεσις or in the demonstration of a previous proposition). But these, like problems, are infinitive phrases rather than what *we* would call propositions. There are also some propositions in Euclid's arithmetical books (the group beginning VIII,8, on how many numbers "fall" in continued proportion between two given numbers, and IX,20, "prime numbers are more than any assigned multitude of prime numbers"), as well as X,1 ("two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out"), which are phrased as theorems but might be stated in modern terms as universal-existential propositions, with the existential quantifier ranging over finite *sequences* of arbitrary length. But Euclid certainly does not understand the propositions in this way, and

problem with the universal-existential proposition, but insisted that such propositions could only be proved constructively; nor am I saying that they *distinguished* the problem from the universal-existential proposition, and said that a narrower range of demonstrations would be acceptable for the problem than for the theoretical proposition. The truth is that neither Greek mathematicians nor Greek philosophers had a conception of a universal-existential proposition at all, and that Greek geometers phrased so many of their propositions as problems, as challenges to do or construct something, partly as an attempt to compensate for the lack of a logic that could handle multiply quantified propositions.

With these caveats, we can try to give a logical description of problematic analysis. In the problem $\forall x\ (Px \rightarrow \exists y\ Qxy)$, on the Greek understanding, the ζητούμενον is not a proposition but an object y such that Qxy. In analysis, we assume a given x such that Px, and we also assume the ζητούμενον y such that Qxy. We then make constructions and inferences from x and y, using the given Px and the assumption Qxy. Eventually we construct an object z having some particular relation to x, such that we can determine z merely from knowing what x is and from knowing that Px, without relying on y or on the assumption Qxy. Using "φ" etc. as symbols for construction-procedures, this happens if from $Px \cap Qxy$ we can derive $Rx\varphi(x, y)$, and if from $Px \cap Rxz$ we can derive $z = \psi(x)$. If this succeeds, we try to reverse the analysis, first by proving that Px implies $Rx\psi(x)$, and then by reversing the construction of $z = \varphi(x, y)$ to give a construction $y = \chi(x, z)$, and proving that $Px \cap Rxz$ imply $Qx\chi(x, z)$. If this can be done, then setting $y = \chi(x, \psi(x))$ gives us a construction-procedure and a proof for $\forall x\ (Px \rightarrow \exists y\ Qxy)$.[15] Or, with problematic as with

indeed the awkwardness of these propositions reflects his lack of the concept of a sequence as the kind of object that can be quantified over. This kind of difficulty seems not to arise in the geometrical books.

[15] It is worth stressing that there is a close formal parallelism between problematic and theoretic analysis, since this may be disguised by the symbolism I have used, which is considerably more complicated for problematic than for theoretic analysis. But we could instead describe problematic analysis by mimicking the simpler description of theoretic analysis, by saying "assume the ζητούμενον $\exists y\ Qxy$ as well as the given Px, then infer a proposition $\exists z\ Rxz$ which we know to be true (or false) on the basis of the given Px, then (if the result was positive) try to reverse by showing that $\exists z\ Rxz$ implies the ζητούμενον $\exists y\ Qxy$". Here, by treating $\exists y\ Qxy$ as if it were a monadic predicate of x, we are putting the problematic analysis into the form of a theoretic analysis. And this is in fact a logically correct description of problematic analysis, as long as we require proofs of existential propositions to be constructive: thus proving

theoretic analysis, we might reach a negative result, discovering that the assumption Qxy contradicts the given Px: this would happen if (as before) we construct some $\varphi(x, y)$ satisfying a relation $Rx\varphi(x, y)$, and then, instead of (as in the positive case) recognizing that $[Px \cap Rxz] \Leftrightarrow [Px \cap (z = \psi(x))]$, we recognize that $Px \cap Rxz$ imply a contradiction, and that the problem is therefore unsolvable. But more often than reaching a *purely* negative result, we will discover that the problem is solvable only under some conditions: this will happen when we recognize that $Px \cap Rxz$ imply some further condition Sx; we can then try to reverse the analysis by showing that, under the hypothesis Sx, z is given by some construction $\psi(x)$, and that $Px \cap Sx$ imply $Rx\psi(x)$, and so on, using $z = \psi(x)$ to construct y and to deduce Qxy. In this case analysis will have revealed the διορισμός, the necessary and sufficient conditions for the problem to be solvable, and this was indeed one major use of problematic analysis in Greek geometry. It may also happen that we do not know how to lead the analysis to either a positive or a negative result, but that we can reduce the problem "given x such that Px, to construct y such that Qxy" to an easier or more fundamental problem, "given x such that Px, to construct z such that Rxz": this will happen if, as before, we find a construction-procedure $\varphi(x, y)$ and prove that $Px \cap Qxy$ imply $Rx\varphi(x, y)$, and if we can then reverse this analysis by finding a construction-procedure $\chi(x, z)$ and proving that $Px \cap Rxz$ imply $Qx\chi(x, z)$. Once again, this was a major use of problematic analysis: indeed, it seems reasonable to describe Hippocrates of Chios' reduction of the problem of doubling the cube to the problem of finding two mean proportionals as an early application of problematic analysis.

Problematic and theoretic analysis are formally similar enough that (in both ancient and modern accounts) they are often covered by the same general description: these descriptions tend to apply more immediately to theoretic analysis, which is logically simpler, leaving problematic analysis as an awkward complication. Nonetheless, it is clear both that problematic analysis is historically older, and that it was heuristically more fruitful. That problematic analysis is older is natural enough, since it is an older and more basic task of geometry to construct or find objects satisfying given descriptions (and the most basic task, that of measuring e.g.

$[Px \cap \exists y\ Qxy] \rightarrow [\exists z\ Rxz]$ means finding a construction-procedure $\varphi(x, y)$ and proving $\forall y\ ([Px \cap \exists y\ Qxy] \rightarrow Rx\varphi(x, y))$; proving $Px \rightarrow [\exists z\ Rxz]$ means finding a construction-procedure $\psi(x)$ and proving $Px \rightarrow Rx\psi(x)$; and proving $[Px \cap \exists z\ Rxz] \rightarrow [\exists y\ Qxy]$ means finding a construction-procedure $\chi(x, z)$ and proving $\forall z\ ([Px \cap Rxz] \rightarrow Qx\chi\ (x, z))$.

an area, was construed as the problem "to construct a square equal to a given area"), while clear standards of proof and justification develop only over time. And while theoretical analysis is essentially a method for discovering a proof of a given proposition, problematic analysis is originally a method for discovering a construction-procedure, although it can also help us discover a proof that this procedure does what it is supposed to. This also helps to explain why problematic analysis was heuristically more important: to apply theoretical analysis to a proposition P, we must already have come to suspect somehow that the proposition is true (one common use would be in teaching, where the student believes P because the teacher [or a book] says so, and then tries to find the proof for himself using analysis).[16] In problematic analysis, on the other hand, while we must suspect that the problem has *some* solution (and often it is intuitively obvious [e.g. by a νεῦσις] that it does, at least subject to some διορισμός), we may have no clue at all about what the solution will be. If we are trying to prove (and prove constructively) a proposition $\forall x\ (Px \rightarrow \exists y\ Qxy)$, problematic analysis can suggest a function φ and suggest that we try proving $\forall x\ (Px \rightarrow Qx\varphi(x))$, and this is likely to get us over the biggest hurdle towards finding the proof; and in this way problematic analysis may lead us to theorems as well as to solutions of problems. All these reasons help to explain why, in the Greek texts, we find mostly *descriptions* of what seems to be theoretical analysis, but mostly *examples* of problematic analysis.

II

Given this description of geometrical analysis, as it was practiced in Plato's time and after, we are in a position to ask: does Plato refer to the method of analysis, and does he (like late ancient Platonists and Peripatetics) take it as a model for philosophical reasoning toward first principles?

As I have already said, Plato never uses the *word* "analysis"; but this is compatible with his being aware of analysis as a distinctive geometrical practice, and with his alluding to this practice without using the name "analysis." In fact it is tolerably certain, not only that Plato was aware of analysis as a distinctive geometrical practice, but also that he knew it under the name "analysis," and that he was familiar with roughly the same

[16] Compare Diogenes Laertius VII,179, where Chrysippus tells Cleanthes that he wants only to be taught the doctrines, and he will find the demonstrations for himself.

(inadequate) logical description of analysis that we find in later writers such as Alcinous and Pappus. The reason I think Plato must have known the name "analysis," and the ancient logical description of analysis, is that we find the name and the description in three passages of Aristotle (two of them plausibly written in Plato's lifetime). Aristotle had no special expertise in mathematics going beyond his Academic colleagues, and the texts show that he is referring to a method that he expects his students, not merely to have heard of, but to be accustomed to practice themselves (the texts are in fact unintelligible except to someone who already knows what analysis is, and have often been misunderstood by both ancient and modern readers). Aristotle is reflecting mathematical knowledge that was current in the Academy, and using it to make his own philosophical points; and the same mathematical knowledge was available for Plato to use in making his philosophical points, if this is what he wanted to do.

I will first briefly go through the texts of Aristotle, to show what knowledge of analysis could be presupposed in the Academy; then I will argue that in at least one extant text – the *Meno* – Plato does refer to the method of analysis, although not by name; then I will comment on the harder question of what philosophical points Plato thought this mathematical practice could illustrate.

Two of the Aristotle passages are in logical contexts, and make roughly the same point: *Posterior Analytics* I,12 78a6-13 and *Sophistici Elenchi* 16 175a26-28. The *Posterior Analytics* passage says: "If it were impossible to show [δεῖξαι = deduce] something true from something false, then analysis [τὸ ἀναλύειν] would be easy: for [the analysis] would necessarily convert [ἀντιστρέφειν]. For let A be true [ὄν]; but if this is true, these things (say, B) are true [i.e. I can deduce B from A], which [sc. B] I [in fact] know to be true. Then from these things I will show that that [sc. A] is true.[17] Now mathematical [arguments] convert more often, because they do not assume something accidental [as a premiss] (and in this they differ from dialectical arguments), but rather [they assume as premisses]

[17] I am translating ὄν throughout as "true", which is the easiest way to take the passage (but NB "true" in the first sentence is the unambiguous ἀληθές); if this is right, Aristotle is talking about theoretical analysis. But it is just possible that "A is ὄν means "A exists"; e.g., if "A" stands for "equilateral and equiangular pentagon," then "A is ὄν" means "there is an equilateral and equiangular pentagon," in which case Aristotle is giving a – rather less logically precise – description of problematic analysis. Nothing much hangs on this; either (as I will assume) Aristotle is talking about theoretical analysis, or he is assimilating problematic and theoretical analysis so closely that it is impossible to tell them apart.

definitions." The passage is very condensed, but Aristotle's basic point is clear enough.[18] Aristotle has been saying that there are valid arguments with (some or all) false premisses but true conclusions. He then illustrates this with an appeal to his readers' (or hearers') experience: life would be much easier in geometry if this were *not* the case, i.e. if every valid argument to a true conclusion also had (all) true premisses, since then an analysis could always be converted into a synthetic demonstration. I am trying to demonstrate a ζητούμενον A;[19] so, for purposes of analysis, I assume A as if it were known to be true, and deduce B, which I in fact know to be true; so the analysis terminates. If every valid argument to a true conclusion had all its premisses true, then, since there is a valid argument from A to B, and since B is true, necessarily A would also be true; so we could automatically convert the argument from A to B into a valid argument from B to A; and since we know that B is true, this would give a demonstration of A. In fact, since some arguments do lead from false premisses to true conclusions, not all analyses convert; but Aristotle adds that, as a matter of mathematical experience, analyses often do convert. I am not sure exactly what to make of Aristotle's explanation, namely that mathematical inferences take definitions rather than accidental properties as their premisses; but one illustration would be that, if an inference (say, that a certain equality holds) depends on the premiss that a certain angle is right, it will use the full strength of the premiss that the angle is right, and not merely that it is (say) greater than 80 degrees; so it is likely to be possible to reverse the inference, to infer that if the equality holds the angle is right, since if the angle was slightly more or less than a right angle, one quantity would be slightly too large or too small. Something

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[18] Themistius misunderstands the passage in his paraphrase (*Analyticorum posteriorum paraphrasis* 26,22-8), apparently due to his ignorance of geometrical practice. Aristotle means: "Suppose we are trying, by the method of analysis, to find a proof that A. We infer from A to B, which we recognize to be true. But this does not yet show us that A is true, because a false premiss could yield a true conclusion." Themistius takes Aristotle to mean: "Suppose we are trying to find a proof that A. We realize that B implies A. But that doesn't yet show that A, because B might be false." The Aristotle text is (like the whole *Posterior Analytics*) highly elliptical, and Themistius fills in the ellipses incorrectly, apparently because he is unfamiliar with the practice of analysis and does not realize that the analyst takes the ζητούμενον (here A) as a premiss in the analytical stage of his argument. Themistius is apparently assuming that analysis must be something like rhetorical *inventio*, a practice with which he is much more familiar.

[19] Assuming that A is a proposition rather than an object. We could rewrite all this in the case where A is an object.

like this is why theorems like the Pythagorean theorem tend to have true converses, although of course this does not hold in every case.[20]

This passage from the *Posterior Analytics* is enough to show that Aristotle, like later Greek writers, conceives of analysis as a process in which a ζητούμενον is assumed to be true, and deductions are made from it until we deduce something independently known to be true; we then try to convert the argument into a demonstration of the ζητούμενον. What may not be clear is what part of this process is called "analysis." Aristotle says that if all arguments converted, τὸ ἀναλύειν would be easy: this suggests that ἀναλύειν is not just what later writers call analysis, namely the process of deducing from the ζητούμενον a proposition known to be true, but rather the whole process that later writers call analysis-and-synthesis, culminating in a demonstration of the ζητούμενον. However, Aristotle's usage is in fact the same as later writers', as is shown by *Sophistici Elenchi* 16 175a26-28: "it sometimes happens as in διαγράμματα ['diagrams' but also 'geometric proofs']: for there too sometimes, after we have analysed [ἀναλύσαντες] we are unable to synthesize [συνθεῖναι] again." Here ἀνάλυσις and σύνθεσις are clearly two successive stages, and the difficulty of passing from ἀνάλυσις to σύνθεσις is what the *Posterior Analytics* passage calls the difficulty of "converting." So when Aristotle says in the *Posterior Analytics* that "it would be easy to ἀναλύειν", he must mean not simply that it would be easy to find an analysis of a ζητούμενον, but that it would be easy to find a *good* analysis, where a good analysis is one that can be converted into a demonstration of the ζητούμενον. (It is always trivial to give some formally legitimate but mathematically useless analysis, and no one would take this as a goal.)

In these passages from the *Posterior Analytics* and *Sophistici Elenchi* Aristotle is interested in drawing analogies between the methods of analysis

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[20] Aristotle's description is inadequate in some of the same ways that other descriptions of analysis, ancient and modern, typically are. He does not distinguish the ζητούμενον from the proposition to be demonstrated. Connected with this, he treats the argument that might or might not convert as simply an inference from one proposition to another, when it is actually embedded in a natural deduction context (i.e. we are arguing from Px to Qx, neither of which is properly speaking a proposition, since they each contain a free variable). Aristotle also speaks as if there were a single premiss A and a single conclusion B, when it would be more accurate to say that we can deduce B from the premiss A and auxiliary premisses C_1, C_2, etc.; the argument is very unlikely to be "convertible" to an argument from B to the conjunction of the premisses $A \cap C_1 \cap C_2$, but it might be convertible to an argument from the conjunction $B \cap C_1 \cap C_2$ to A.

and synthesis in geometry and procedures of non-mathematical reasoning, although it is not clear that he is willing to use the word ἀναλύειν in non-mathematical cases. However, in a third passage, Aristotle does use "analysis" metaphorically to describe, not philosophical inferences, but practical reasoning from ends to means.

> Having posited the end, they examine how and by what means it will come about: and if it seems that it can come about in many ways, they also examine which is the easiest and best, but if it is accomplished [only] in one way, they examine *how* it will come about through this means and how this means itself will come about, until they come to the first cause, which is last in discovery. For the person who deliberates seems to be inquiring and analyzing [ζητεῖν καὶ ἀναλύειν] in the way that has been described, as if [he were inquiring into and analyzing] a διάγραμμα (it seems that not all inquiry [ζήτησις] is deliberation – mathematical ones are not – but all deliberation is investigation); and the last thing in the analysis is the first in the coming-to-be. And if they encounter something impossible, they desist, for instance if money is needed and there is no way to provide this; but if it seems possible, they try to do it. (NE III,3 1112b15-27)

This passage is difficult, but it is clear that Aristotle is thinking specifically of *problematic* analysis, and using it as a model to describe practical reasoning: we begin with a specification of the object we are trying to produce, and, positing a situation in which this has been achieved, we reason back to the way it might have been produced, until we reach something that is immediately in our power to produce. This last thing corresponds, in a problematic analysis, to the last thing we construct from the ζητούμενον, which we recognize as something that is determined by the data of the problem, so that we are able to construct it directly from the data: so this "last thing in the analysis" is the "first in the coming-to-be" of the ζητούμενον; then, in reversing the analysis, we construct each of the subsequent things out of this first thing, in the reverse of the order in which we found them in the analysis, until we have constructed the ζητούμενον. The analysis has succeeded only when we have inferred from the ζητούμενον back to a "first cause" or ἀρχή, meaning not a proposition we know to be true, but an object we know we can construct; whereas, if we infer to an object related to the givens of the problem in an impossible way, we have a *reductio ad absurdum*, and we give up the problem as unsolvable in the given case.[21]

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[21] Aristotle may well also be thinking of analysis at NE VI,8 1142a23-30, where φρόνησις (the ability to deliberate well), which perceives some ἔσχατον that is πρακτόν, is compared to an ability to perceive that τὸ ἐν τοῖς μαθηματικοῖς ἔσχατον is (for instance) a triangle. Τὸ ἐν τοῖς μαθηματικοῖς ἔσχατον *might* mean simply a

These passages from Aristotle show that it was possible to presuppose a familiarity with the practice of geometric analysis in Academic circles (in Plato's lifetime and in the subsequent decades); that the practice was known by the name "analysis," and that essentially the same (inadequate) logical description of analysis that we find in Pappus was already available; and, finally, that Academic philosophers were interested in using geometrical analysis, so described, as a model for philosophical (and practical) reasoning. So, even though Plato never uses the word "analysis," he and his students in the Academy were familiar with the practice; Plato could (if he wanted) allude to this geometric practice (expecting his Academic readers to fill in the name "analysis" and the logical description), and make whatever point he might want to make about the relation between this kind of mathematical reasoning and reasoning in philosophy. I will now argue that Plato does, once, so allude to analysis, in the second geometrical passage of the *Meno* (86e4-87b2); and then I will offer some speculations about what philosophical point Plato wanted to make by the analogy with geometry.

Socrates has proposed to examine "from a hypothesis" Meno's question whether virtue is teachable. He then says,

> I mean "from a hypothesis" in this way, the way the geometers often examine, when someone asks them, for example, about an area, whether it is possible to inscribe this area in this circle as a triangle. [A geometer] might say, "I don't yet know whether this [area] is such [as to make the construction possible], but I think I have as it were a hypothesis that would help towards the question, as follows: if this area is such that when it is applied to the given line [sc. the diameter of the circle], it falls short by an area similar to the applied area, then one thing seems to me to follow, but another if it is impossible for this to happen. So after hypothesizing I am willing to tell you what follows about inscribing [the area] in the circle, whether it is impossible or not."

Here Plato is considering a geometrical problem, "to inscribe in a given circle a triangle equal to a given area"; in fact the hypothesis he gives is designed to solve the more specific problem "to inscribe in a given circle an *isosceles* triangle equal to a given area."[22] The hypothesis that Plato

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mathematical particular, though it seems odd to posit a special quasi-sensory ability to recognize (individual, but perfect and hence non-sensible) mathematical triangles; but it seems more likely that τὸ ἐν τοῖς μαθηματικοῖς ἔσχατον is the last thing constructed in an analysis, which we quasi-perceptually recognize as something we already know how to construct from the givens.

[22] The special problem is equivalent to the general problem in the loose sense that whenever there is a solution to the general problem there is also a solution to the

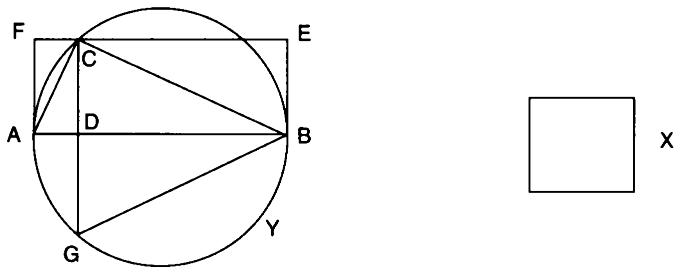

Figure 1

mentions, namely that the given area can be applied to the diameter of the given circle (in the form of a rectangle) in such a way that it falls short by a figure similar to the applied area,[23] is in fact a necessary and sufficient condition for the problem to have a solution; furthermore, any solution to the application-of-areas problem can be straightforwardly converted into a solution of the problem "to inscribe in a given circle an isosceles triangle equal to a given area." For (see Figure 1) let AB be a diameter of the given circle Y, and let the rectangle CDBE be equal to the given area X, and let the rectangle CDBE fall short of the line AB by the rectangle FADC, in such a way that the rectangle CFAD is similar to the rectangle CDBE. Thus the line CD is a mean proportional between the line AD and the line BD. Produce the line CD beyond D to G, so that GD = CD. Since the rectangle on GD and CD (being equal to the square on CD) is equal to the rectangle on AD and BD, it follows (by the converse of Euclid III,35) that the points A, B, C and G lie on a circle. Since the chord AB perpendicularly bisects the chord CG, AB must be a diameter, so the circle on which the points A, B, C and G lie is in fact the given circle Y. Now the triangle CDB, which is half of the rectangle

special problem. It is not equivalent in a stronger sense, since there is no straightforward procedure for converting a solution to the general problem into a solution to the special problem.

[23] An area X is *applied* to the line AB in the form of a rectangle (or parallelogram) if a rectangle (parallelogram) equal to X is constructed with AB as base; the applied figure *exceeds* AB if its base is the line AC which extends AB to C lying beyond B (and it exceeds AB *by* the portion of the figure that lies over BC); it *falls short of* AB if its base is AC for C lying in between A and B (and it falls short of AB *by* the portion of the figure that lies over BC).

CDBE, is also half of the triangle CGB. So CGB is equal to CDBE, which is equal to the given area X. So CGB is an isosceles triangle inscribed in the given circle Y and equal to the given area X: which is what was to be found.

Plato explicitly cites this example, not as an example of analysis, but only as an example of how a geometer might reason from a hypothesis in answering a given question. The question he cites proposes a problem rather than a theorem, and a full answer would be a solution to the problem: that is, the aim is not simply to answer the question "is it possible to inscribe this area in this circle as a triangle?" with yes or no, but rather, in the case where the answer is yes, to give a construction-procedure showing *how* to inscribe the area in the circle in the form of an [isosceles] triangle.[24] When the geometer answers the question "from a hypothesis," he is taking a step toward answering the question fully, that is, toward giving both a διορισμός for the problem and a construction-procedure for solving it where it can be solved. So when the geometer offers to answer the question from the hypothesis "the given area can be applied to the diameter in such a way that it falls short by a figure similar to the applied area," he is not simply claiming that the original problem is solvable if and only if the application-of-areas problem is solvable, but also offering a construction-procedure to convert any solution of the application-of-areas problem into a solution to the original problem. The solution "from a hypothesis" thus *reduces* the original problem to the application-of-areas problem: the task that remains is to give a διορισμός determining whether the application-of-areas problem can be solved for the given area and the given line, and to give a construction-procedure for solving it where it can be solved. When Plato recommends the geometers' practice of answering "from a hypothesis," he is recommending tackling a difficult question by reducing it step-by-step to more basic questions until we can answer it directly: and this is the lesson Socrates draws when, in answering Meno's question "is virtue teachable?" from the hypothesis "virtue is knowledge," he *reduces* Meno's question to the question "is virtue knowledge?" (87b2-d1), then answers this question in turn from the hypothesis "virtue is good" (87d2-89a7; explicitly called a "hypothesis" at 87d3), which presumably we can immediately grasp to be true. So Plato is recommending, not simply that we learn how to answer a given question X

[24] If the only interest were in giving a διορισμός, this would be much easier: the answer would be "if and only if the given area is less than or equal to an equilateral triangle inscribed in the given circle."

from a given hypothesis Y, but also that we learn how to tackle a given question X by *finding* an appropriate "hypothesis" to reduce it to.[25] In giving the geometrical example, Plato leaves it mysterious how the geometer finds the appropriate hypothesis: on a superficial reading, it looks as if the geometer is simply guessing, or intuitively divining that the hypothesis "the given area can be applied to the diameter in such a way that it falls short by a figure similar to the applied area," would be useful for investigating the problem at hand; it would then be just a lucky coincidence, or a confirmation of the geometer's power of intuition, that the hypothesis turns out to be necessary and sufficient for solving the problem. But in fact this hypothesis was certainly found by the method of analysis, and is very typical of the use of analysis in reducing a problem to an easier problem; and since Plato is recommending a method for finding appropriate hypotheses and so reducing hard questions to easier ones, it is analysis that he is recommending.

To see how analysis of the problem "to inscribe in a given circle Y an isosceles triangle equal to a given area X" would lead to Plato's hypothesis, assume the problem solved. So (see Figure 1) let BCG be an isosceles triangle, BC = BG, inscribed in the circle Y and equal to the rectilineal area X. Then let BA be a diameter of the circle Y; the diameter BA perpendicularly bisects the chord CG at a point D. Connect AC. The angle ∠ACB is inscribed in a semicircle, and is therefore a right angle. So the triangles ADC and CDB are similar, to each other and to the triangle ACB. So, completing the rectangles ADCF and CDBE, we see that these rectangles are similar, and therefore that the rectangle CDBE falls short of the line AB by a figure similar to itself. Since the rectangle CDBE is double the triangle CDB, which is half of the triangle BCG, it follows that CDBE = BCG; but BCG = X, so CDBE = X. So the given area X has been applied to a diameter of the given circle Y in the form of a rectangle, in such a way that it falls short of the diameter by a figure similar to the applied area. As we have seen, the analysis can be reversed, so the hypothesis "the area X can be applied to the diameter of Y in the form of

[25] In *Prior Analytics* II,25, Aristotle gives this passage (without citing the *Meno* by name) as an example of reduction [ἀπαγωγή]: we wish to know whether teachable belongs to virtue, it is clear that teachable belongs to knowledge, so we reduce the question whether teachable belongs to virtue to the (hopefully) simpler question whether knowledge belongs to virtue. Aristotle compares this to a geometrical example, Hippocrates of Chios' attempt to reduce the problem of squaring the circle to simpler problems.

a rectangle in such a way that it falls short by a figure similar to the applied area" gives a sufficient as well as a necessary condition for solving the problem "to inscribe in the circle Y an isosceles triangle equal to X."

This is not only an easy and straightforward use of analysis, but also a very typical one. In fact, it seems to be part of a systematic program of reducing problems of all kinds to problems of application of areas, in the hope that these problems could all be solved in a simple and uniform way. An important example is the problem of constructing a regular pentagon, which can be reduced by analysis to the problem "to divide a line in extreme and mean ratio"; this in turn can be reduced by analysis to the problem "to apply a square to its own side in the form of a rectangle in such a way that it exceeds by a square," and this in turn can be reduced by analysis to the problem of finding a mean proportional, and thus solved. Probably beginning from the analysis of the regular pentagon, early Greek geometers developed techniques for solving a broad class of problems of application of areas: Proclus (*In Euclidem* 419-20) cites Eudemus as attributing these techniques to "the Muse of the Pythagoreans" (i.e. to the tradition from Hippasus to Archytas) and Euclid presents their results in developed form in *Elements* VI. The original problems would have been "to apply a given area to a given line, in the form of a rectangle, in such a way that it exceeds [or falls short] by a square," but the techniques for solving these problems can be generalized to solve "to apply a given area to a given line, in the form of a rectangle, in such a way that it exceeds [or falls short] by a rectangle similar to a given rectangle" or even "to apply a given area to a given line, in the form of a parallelogram, in such a way that it exceeds [or falls short] by a parallelogram similar to a given parallelogram," which is the problem that Euclid solves in *Elements* VI,28-29. Euclid gives a διορισμός for the problem of falling-short (which cannot be solved in all cases), and gives a construction which works by reducing both problems to the problem of constructing a parallelogram of a given shape with a given area, which in turn can be reduced to the problem of finding a mean proportional; while Euclid does not explicitly give the analyses of his application-of-areas problems, his exposition makes it obvious that his constructions (and his διορισμός) were first discovered by analysis.[26] The application-of-areas problem that Plato proposes as his

[26] In fact the proof of VI,27, giving the διορισμός for the falling-short problem VI,28, is a disguised analysis (and thus, in a sense, the earliest extant analysis). What Euclid is doing in these propositions seems to arise from a generalization of the results needed to construct the regular pentagon. Euclid draws from VI,29, the problem of

"hypothesis" in the *Meno* is obviously similar in formulation to Euclid's problems, and comes from the same geometrical research-program; but Plato's problem is more difficult, and Euclid does not discuss it in the *Elements* because it cannot be reduced to finding a mean proportional or solved by ruler-and-compass constructions. However, there is a solution using conics, which Plato may well have known when he wrote the *Meno*, and which was certainly within the capacity of Greek geometers at least least by mid-fourth-century.[27] But whether Plato knew the solution or not, he would have seen the problem as part of a promising program for finding διορισμοί of any given construction-problem and for solving any problem when it can be solved.[28]

Thus Plato alludes at *Meno* 86e4-87b2 to the method of analysis, and more specifically to the program of reducing construction-problems through problematic analysis; and he holds up the program of analysis as a methodological model for philosophical inquiry. But Plato does all this without ever using the word "analysis" (though he must have known the word), and without describing clearly either the logic of the method in general or the geometry of the case he describes: he does not explain either how his application-of-areas problem was derived from the original problem, or how it would help to solve the original problem, or how it might itself be solved; indeed, he does not describe either the original problem or the "hypothesis" clearly enough for anyone who did not already understand the problem Plato is describing to understand him.[29]

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excess, the corollary VI,30, "to divide a line in extreme and mean ratio." Euclid does not use VI,30 to construct the regular pentagon, because he has already done it in IV,10-11 without using proportion theory, using II,11, "to divide a straight line so that the rectangle contained by the whole and one segment is equal to the square on the other segment": but this is simply VI,30 reformulated, and re-proved, in such a way as to avoid proportions. Euclid is certainly modifying an earlier order of presentation which used application of areas to construct the pentagon.

27 See the solution of the *Meno* problem given by Heath, *History of Greek Mathematics* (Oxford, 1921), v. 1, pp. 300-301, using the same methods that Menaechmus (a student of Eudoxus) used to find two mean proportionals between two given lengths (see Heath, v. 1, pp. 251-5).

28 In the passage of Philodemus' *Academica* (ed. Gaiser, Stuttgart, 1988, p. 152) reporting some Academic or Peripatetic source on the progress of mathematics in the Academy, and speaking of Plato as research-director, it is said that "analysis and the taking of διορισμοί [ἡ ἀνάλυσις καὶ τὸ περὶ διορισμοὺς λῆμμα]" were then brought forth; the conjunction is apparently the subject of a singular verb.

29 Hence the gross misunderstandings of this passage e.g. in Jowett's and Grube's translations, and by Bluck in the appendix to his edition of the *Meno* (Cambridge,

This compressed passage, alluding without explanation to what would have been the forefront of mathematical research, immediately after an extremely slow and patient discussion of a trivial mathematical fact, was not meant to be understood by most of its readers; Meno, who asks no questions at all at this crucial turning point in the argument, obviously does not understand what is going on. In fact, the passage is perfect proof for Gaiser's thesis that Plato's dialogues allude to doctrines that they do not fully explain, in an attempt to rouse Plato's readers to seek further enlightenment in the Academy.[30] Those of Plato's readers who are familiar with current geometric practice will understand his mathematical allusions; his other readers will pick up that Plato is referring to *some* geometric result and to *some* geometric practice which is supposed to be philosophically important and which they would understand if they came to study geometry in the Academy.[31] But what was the philosophical payoff supposed to be?

III

As I said at the beginning, analysis appealed to philosophers because it presented a method for *discovery*. In the *Meno*, the progress of inquiry had been frustrated: Socrates cannot answer Meno's question "is virtue teachable" until Meno tells him what virtue is, and Meno, unable to define virtue after repeated attempts, gives what Socrates calls an "eristic argument" (80e2) to show that one cannot search for something if one does not already know what it is. So Socrates' immediate task is to show Meno how inquiry is possible, by giving him successful models of it: this is the point of both the first and the second geometrical passages. The first geometrical passage, together with the account of immortality and recollection which it illustrates, helps to show how we can inquire "what is X"

1964); but the passage is correctly translated e.g. by Guthrie, and by Heath, *History of Greek Mathematics*, v. 1, p. 299, followed in the revised version of Grube's translation in John Cooper and D.S. Hutchinson, eds., *Complete Works of Plato* (Indianapolis, 1997).

[30] I am not endorsing Gaiser's view of the *content* of these doctrines.

[31] Compare Heath, *History of Greek Mathematics*, v. 1, p. 302. Benecke had objected that, on the (correct) interpretation favored by Heath, Socrates would be describing quite a difficult geometrical problem, and that therefore "Plato is unlikely to have introduced it in such an abrupt and casual way into the conversation between Socrates and Meno"; Heath replies, rightly, that "Plato was fond of dark hints in things mathematical."

(e.g. what is virtue?): we have encountered X before this life and so have dim memories of it, which will help us to recognize the thing when we are confronted with it again, and which can be teased out and "tied down" to become knowledge. The second geometrical passage serves a complementary function in showing how inquiry is possible. Socrates had wanted Meno (encouraged by the account of recollection) to keep on inquiring "what is virtue?" (86c4-6), but Meno wants to go back to his original question whether virtue is teachable or acquired in some other way (86c7-d2), that is, to ask *what virtue is like* [ποῖόν ἐστι] before determining *what it is* [τί ἐστι]. Surprisingly, although Socrates had earlier insisted that it was impossible to inquire in this way, he now immediately gives in, and offers to investigate Meno's ποῖόν ἐστι question on the basis of a hypothesis about the τί ἐστι. As we have seen, this hypothetical investigation means using something comparable to the method of analysis to *reduce* the ποῖόν ἐστι question to a τί ἐστι question, and to keep reducing it until we reach a question that we can answer directly. In a sense, Socrates has not conceded much on the logical priority of the τί ἐστι to the ποῖόν ἐστι question, since he continues to insist that we cannot *know* whether virtue is teachable until we can demonstrate the answer from a knowledge of *what virtue is*. But as a matter of heuristics, Socrates is conceding that it may be useful to begin with the logically posterior ποῖόν ἐστι question, in the hope of discovering an answer both to the τί ἐστι and to the ποῖόν ἐστι questions. Certainly Meno had not been making much progress in his successive attempts to answer the τί ἐστι question directly, so perhaps it is worth trying an indirect approach. The geometers are supposed to be able, through analysis, to reason from logically posterior things to logically prior things, and so to discover the appropriate principles for demonstrating an answer to a given question; so perhaps we can imitate their success in philosophy. Thus the first geometrical passage suggests that it is in principle possible come to knowledge of what virtue is, if someone can discover the right series of questions to ask; the second passage suggests that something like an analytic investigation of whether virtue is teachable might be the path that succeeds, in bringing us to knowledge of whether virtue is teachable, and thus also of what virtue is.[32]

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[32] If Plato had thought it was worth while, he could also have illustrated the method of analysis in the *first* geometrical example, by showing how the line of questioning that prompts recollection is an application of the method of analysis. In one sense he is in fact doing this, since part of what prompts recollection is the refutation of the answers that the side of the eight-foot square is four or three; and such a *reductio ad*

What is much less clear is how this is supposed to work. If we start by not knowing the ἀρχαί that we need for demonstrating the answer to our question (here "is virtue teachable?"), how is analysis or its philosophical analogue going to help us find the demonstration? Later Greek philosophers identify analysis with arguing "upward" to the ἀρχαί, and suggest that we can first argue "up" from posterior things to the ἀρχαί, then argue back "down" from the ἀρχαί to the posterior things. But this kind of argument will not give a *demonstration*, and so will not give us *knowledge* of the posterior things, unless we have acquired *knowledge* of the ἀρχαί: and how is analysis or its analogue supposed to help in that?

There are a number of different senses in which analysis could be said to lead to knowledge of ἀρχαί, and it will help to sort some of these out. To begin with, ἀρχή can be taken as equivalent to ὑπόθεσις, as the proposition which is "laid down" at the beginning of a discourse, to fix the reference of a term or to give a premiss for a deduction.[33] In this sense, when

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absurdum is just an analysis with a negative result. However, the positive result that the side of the eight-foot square is the diagonal of the four-foot square is not shown as being reached analytically: if Socrates had not already known the answer, and asked the boy the right string of questions based on knowledge of this answer, the boy might never have discovered it. But Plato could instead have shown this result as being reached by analysis: suppose a square of area eight square feet has been found; draw the diagonals, dividing the square into four equal isosceles right triangles, each of which is thus of area two square feet. At this stage, probably, something clicks, and we recognize that half of the given two-foot-by-two-foot square, divided by a diagonal, is also an isosceles right triangle of area two square feet. We thus know how to construct purely from givens a figure similar and equal to the figure we have constructed from the ζητούμενον, namely the isosceles right triangle whose base is the side of the ζητούμενον square. And we can then reverse the analysis to construct the ζητούμενον square of area eight square feet from the isosceles right triangle of area two square feet, by constructing four equal and similar isosceles right triangles around the same vertex. The diagram that would result is the diagram that Socrates in fact draws, and Plato could have represented it as the result of reversing this analysis. But Plato probably thought that the method of analysis was too important to waste on such a trivial example: its power is better brought out by showing how it can contribute to a difficult problem belonging to current or recent mathematical research.

[33] Carl Huffman gives a useful collection and discussion of evidence on the early history of the terms ἀρχή and ὑπόθεσις, especially in Hippocratic texts, in the introduction to his *Philolaus of Croton: Pythagorean and Presocratic* (Cambridge, 1993), pp. 78-92. The noun ὑπόθεσις is a relatively late development from the phrase ὑποτίθεσθαι ἀρχήν, "to lay down a beginning" for a discourse, where it is often assumed that the appropriate beginning must be something that the listeners will agree to. The sense "beginning of a discourse" connects with the physical sense of ἀρχή, since often the appropriate beginning for the discourse will be the "natural" beginning

problematic analysis discovers the διορισμός for a given problem, it is discovering an ἀρχή for the proposition, that is, a hypothesis from which the proposition can be proved. But this is not discovering an ἀρχή in such a way that the ἀρχή is *known to be true*: and while I think *part* of what Plato is interested in is simply discovering an appropriate hypothesis for proving a given proposition, he also wants more than that. Thus the *Phaedo* speaks of going from a hypothesis to a higher hypothesis "until you come to something sufficient" at which you can stop (102e1); the *Republic* says that dialectic proceeds from a hypothesis ἐπ' ἀρχὴν ἀνυπόθετον (510b6-7), i.e. to an ἀρχή which is *not* a hypothesis, but which is somehow immediately *known* (and not merely assumed) to be true. If a hypothesis is something like the διορισμός of a problem, or more generally any condition of a proposition that can be reached by analysis, then what is an ἀρχή that is *not* a hypothesis? A look at the *Meno* example suggests why the "hypothesis" there is insufficient, and what a more sufficient ἀρχή might look like. Recall that the problem was posed with regard to a particular area and a particular circle, "whether it is possible to inscribe *this* area in *this* circle as a triangle": the geometer says that he can do it "if this area is such that when it is applied to the given line [sc. the diameter of the circle], it falls short by an area similar to the applied area," but he does not know whether this hypothesis holds. This is because the hypothesis is itself a difficult problem (or says that a problem can be solved), and there is no direct way to verify whether it holds of the given area. By contrast, if the hypothesis were "this area is smaller than another given area," there would be a direct way to check whether it holds (assuming both areas are given as rectilineal figures): if it holds, it can be established, not by a general proof, but by a construction that must be verified by direct perception of *this* particular given area. It seems reasonable to say that when a geometric proposition has been reduced to something that we can verify by direct perception of the given figure, then it has been reduced to an ἀρχή that is not a hypothesis. But this depends on visual perception of a figure: this is no help in dialectic, which, unlike geometry, makes no use of visual images, and can lead us to knowledge only by reasoning. How can reasoning lead us to knowledge of an ἀρχή that is not a hypothesis?

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of the thing. The beginning of the discourse may also be something like a definition, not so much as a starting point for deduction as to make sure the speaker and listeners are talking about the same subject (so too in Plato, *Phaedrus* 237b7-d3). Plato can use ἀρχή and ὑπόθεσις as equivalent, as with ὑπόθεσις at *Phaedo* 101d7 and ἀρχή at 101e2.

I don't have a fully adequate answer to this question, and I don't think Plato did either. But further reflection on geometrical analysis will shed some light on how Plato thought its philosophical analogue might work. To begin with, there is an obvious sense in which analysis is reasoning back to an ἀρχή – not simply to a hypothesis (or to the διορισμός of a problem) but to an ἀρχή that is *known* to be true. In (say) the theoretical analysis of a theorem $\forall x\ (Px \rightarrow Qx)$, we reason from the ζητούμενον Qx (together with the "given" Px and the principles of geometry) back to something known to be true, where this could be either an ἀρχή of geometry absolutely, or an ἀρχή relatively to this particular proposition, that is, one of the givens of the proposition (Px or – since Px is typically a conjunction – one of the conjuncts in Px), or something that has already been deduced from some combination of these ἀρχαί. In any of these cases it is fair to say that we are reasoning back from the ζητούμενον to an ἀρχή that is known to be true.[34] But, since the argument begins by assuming a ζητούμενον which is not (at the outset) known to be true, the inference from the ζητούμενον to the ἀρχή cannot be the *cause* of our knowing the ἀρχή to be true. One possible way out would be to say that we may begin the "upward way" from a ζητούμενον which we "know" to be true through sense-experience or from authority, but which we do not know *scientifically*, because we don't understand *why* it's true. Indeed, in geometry we often start by *believing* that a theorem is true, on the authority of a teacher or of a book, and then apply theoretical analysis in order to discover a proof, and so to understand *why* the theorem is true. In such a case, we begin with a ζητούμενον which we "know" in a weak sense, reason up to an ἀρχή which we know, and reason back down to the ζητούμενον, so coming to know it in a stronger sense; Plato would call our initial state "true opinion" rather than knowledge, and he would describe the whole process as converting true opinion into knowledge by "tying it down" through "reasoning out the cause" (*Meno* 98a3-4). But again, if we have no means of recognizing the truth of the ἀρχή independently of the ζητούμενον, this process cannot give us scientific knowledge: it will leave us with only true opinion of the ἀρχή, and so with only true opinion of

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[34] Plato seems not to be interested in the "logical direction" of analysis, i.e. the fact that starting from the ζητούμενον Q, we work back to a principle or a given P such $Q \rightarrow P$, in the hope of proving, when we reverse the analysis, that $P \rightarrow Q$. Plato speaks as if we just divined P as a plausible starting point for proving Q, and established that $P \rightarrow Q$; but Plato thinks that we also examine the consequences of the hypothesis $\neg P$, proving $(\neg P) \rightarrow (\neg Q)$, and thus indirectly proving $Q \rightarrow P$.

the ζητούμενον. This cannot be what Plato means by Socrates' explanation of how we can arrive at knowledge in philosophy. Indeed, if we take the comparison with geometrical analysis seriously, it rules this out, since the success of analysis depends on our inferring to something that we *already* know to be true, independently of the analytical chain of inferences that lead us to it.

But we need to draw a distinction. An analysis terminates when we succeed in inferring something that we already *habitually* know, but we need not *actually* know it before we infer it from the ζητούμενον. Here I am using Aristotle's terminology of actual and habitual knowledge: I am *actually* knowing a theorem if I am currently thinking about the theorem and understanding why it is true; I have *habitual* knowledge of the theorem if I am in such a state that, whenever I turn my attention to the theorem, if nothing obstructs me from thinking about it, I will understand why the theorem is true. Thus someone who has mastered elementary geometry always has *habitual* knowledge of a large number of theorems, although most of the time he will not have this particular theorem or its proof present to his mind. For an analysis to succeed, we must, when we make the final inference, recognize that its conclusion is something we know to be true: this means that we must already have had *habitual* knowledge of the conclusion, and so the analytical inference itself cannot be the cause of our habitual knowledge of the conclusion, but it may very well be the cause of our *actual* knowledge of the conclusion; that is, it may be the occasion that turns our attention to this proposition, actualizes our habitual knowledge of it, and begins the chain of actualizations (as we reverse each step of the analysis) which leads to our having actual knowledge of the theorem we were trying to prove.

Indeed, it is in some sense *necessary* that, when I am in the process of doing an analysis, I do not yet have actual knowledge of the proposition (already habitually known) in which the analysis will terminate. Suppose I am doing a theoretical analysis of the theorem $\forall x\ (Px \rightarrow Qx)$. Suppose that, in the analytical chain of inferences beginning from the ζητούμενον Qx (and also assuming the given Px), the final proposition I reach is Rx; when I reach the conclusion Rx, I recognize that I already have habitual knowledge that $\forall x\ (Px \rightarrow Rx)$, and therefore, using the given Px, that Rx. Now suppose that the next-to-last proposition I reach in the analysis, immediately before Rx, is Sx. If the analysis is step-by-step reversible, then, as the first step in reversing the analysis, I will be able to prove $\forall x\ (Px \cap Rx \rightarrow Sx)$; since I can also prove $\forall x\ (Px \rightarrow Rx)$, I can prove $\forall x\ (Px \rightarrow Sx)$. But, clearly, I did not have actual knowledge of the proposition

$\forall x\ (Px \rightarrow Sx)$ before I drew the analytic inference from Sx to Rx; for, if I had, I would have stopped the analysis at Sx, rather than going on to Rx. So, even if I had actual (and not merely habitual) knowledge of $\forall x\ (Px \rightarrow Rx)$, that knowledge must have been somehow "obstructed" and prevented from producing actual knowledge of its consequences. If I had not been thus obstructed somewhere down the line, I would have seen right down the chain of consequences from Px to Qx, and so I would have known the theorem immediately, without having to apply analysis to discover a proof. So analysis, if it succeeds, has the psychological effect of removing an obstruction from some of my habitual knowledge, to allow it to have its full consequences in actual knowledge.

Geometrical analysis can thus provide Plato with a model for philosophical discovery, in one sense of "discovery": it does nothing to explain a transition from not having habitual knowledge to having habitual knowledge, but it helps to explain the transition from having merely habitual knowledge to having actual knowledge, that is, the process of removing an obstruction from our habitual knowledge. But, after all, this is all we can expect from Plato, since he renounces the possibility of explaining the first kind of transition. The point of the account of learning as recollection is just to give up on this, and to say that we have always had habitual knowledge, but that it has been somehow obstructed, and that we "learn" by removing obstructions and reawakening the habitual knowledge that is under the surface of our minds. The two geometrical passages of the *Meno* serve complementary functions in explaining how we can come to have actual knowledge: the first argues that we have always had habitual knowledge, and the second uses the model of geometrical analysis to explain how we can go from habitual to actual knowledge.

Analysis infers from a ζητούμενον, assumed but not known to be true, to some kind of ἀρχή already habitually known to be true; we come to actually know the ἀρχή, and thus it becomes available as a starting-point for demonstrating the ζητούμενον. Analysis is designed to bring some possible ἀρχή to our attention, and also to bring to our attention a possible series of inferences from this ἀρχή through intermediate propositions to the ζητούμενον; of course, this can occasion our discovery of an actual proof only if we do have habitual knowledge that the ἀρχή is true and that each of the inferential steps is justified. If we begin from a true opinion of the ζητούμενον (based, perhaps, on the authority of a competent teacher), we have good reason to hope that we will reach a true and usable ἀρχή, thus stimulating recollection of something we already habitually knew but did not have present to our minds; and, as Plato says, "since all nature is akin, and the soul has learned all things, nothing prevents someone, once he has recollected just one thing – and this is what people call learning – from finding out all the others" (*Meno* 81c9-d3). Nonetheless, there is an important disanalogy between geometrical analysis and the kind of philosophical inquiry that Plato wants it to illustrate. In geometry, we are interested in awakening actual knowledge of the ἀρχή only as a means to discovering a proof of the ζητούμενον; the ἀρχή (in the example I have been using, Rx, or $\forall x\ (Px \rightarrow Rx)$) is not in itself something especially desirable to know – it is, generally, an obvious fact but one that had not occurred to us in this connection, or had not seemed useful as a starting point for proving the ζητούμενον. As Plato sees it, the philosophical case is different: although a particular inquirer (such as Meno) may be more interested in the posterior question (whether virtue is teachable) than in the prior question (what virtue is), so that in a particular dialectical situation we may be led to ask about the ἀρχή for the sake of knowing the ζητούμενον, nonetheless Plato thinks that the knowledge of the ἀρχή (of what virtue is, and ultimately, of the good) is intrinsically much more desirable than all the knowledge we can derive from it. The knowledge of the ἀρχή is a great good, but it is one we already have, deep within us; but like the food and drink of Tantalus (apparently recalled at *Euthydemus* 280b-d), it is a possession that we are prevented from using, and so does not actually benefit us. Since this knowledge lies deeply buried within us, to uncover it and make it available would be a great good; whereas the principles of geometry lie pretty close to the surface, and the great thing is not to dig them up but to build something with them. Despite this difference between the aims of geometry and of Platonic philosophy, Plato finds the method of analysis an encouraging model for what he hopes can happen in philosophical discovery. In the *Seventh Letter* he says that the knowledge he aims at "suddenly, like a light kindled from a leaping fire, comes to be in the soul and then nourishes itself" (341c7-d2); but in this same passage he is warning against false claims of insight, and insisting that the leaping and kindling come about only "with the maximum of practice and much time" (344b2-3),[35]

[35] There is an untranslatable pun in τριβή: "practice" in the sense of repeated exercise as opposed to theoretical instruction (perhaps as a way of learning to apply the instruction, but *Gorgias* 463b2-4 describes rhetoric and relish-making as "not τέχνη but ἐμπειρία καὶ τριβή"); also just "spending time" (like διατριβή, the standard word for philosophical education through conversation and companionship); but also "rubbing," a sense which Plato makes good use of here, with a suggestion of starting a fire by friction.

through patient and rigorous inquiry. What Plato is describing is an everyday experience in geometry. When analysis succeeds, something suddenly happens, a spark jumps, we suddenly understand something or see something in a new light, and see how to find what we were looking for; but analysis is also a precise discipline that we can become trained in, and that must be practiced rigorously and patiently for the result to come about; and it is accompanied by a rigorous method of synthesis for checking and for discarding false inspirations. One can only wish there was something like that in philosophy.[36]

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[36] Long after writing the above, I discovered the following passage in Galen, expressing a similar judgment: "Οὐδ' ἄλλη τις θεωρία μειζόνως εὐφραίνει τὴν ψυχὴν ἀνδρὸς εὐφυοῦς τῆς ἀναλυτικῆς, ὅταν γέ τις ἐν αὐτῇ προέλθῃ· κατ' ἀρχὰς μὲν γὰρ ἐπίπονός ἐστιν, ὥσπερ καὶ αἱ ἄλλαι σχεδὸν ἅπασαι. καίτοι κἂν εἰ μηδεμίαν εὐφροσύνην εἶχε, δι' αὐτό γε τὸ μέλλειν αὐτῇ χρῆσθαι πρὸς τὰ μέγιστα καλῶς ἂν εἶχεν ἀσκηθῆναι κατ' αὐτὴν ἐξαίρετον ἔχουσαν, ὡς ἔφην, τὸ μαρτυρεῖσθαι πρὸς αὐτῶν τῶν εὑρημένων, ὅπερ οὐκ ἔστιν ἐν τοῖς κατὰ φιλοσοφίαν εὑρισκομένοις" (*On the errors of the soul*, 5,87,14-88,6 Kuehn; repunctuating following Marquardt and De Boer).

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